Math 102

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October 23, 2018

Announcements

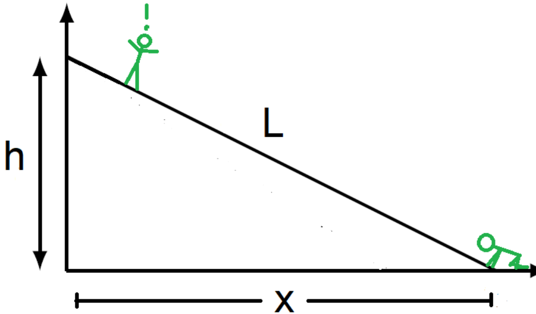
- Review Sessions for Thursday 10/25 Midterm
  - Monday 10/22 in Buchanan A201, 3-7pm
  - Tuesday 10/23 in CHBE 101, 3-7pm
  - Bring questions if you have them!
- Midterm covers material up to and including Optimization.
- See the 'Midterm' tab on Canvas.

Goals Today

- The Chain Rule Applied: Related Rates
- Implicit Functions and Differentiation

Related Rates: Sliding Ladder

A ladder of length L meters is propped against a wall. The base of the ladder is pulled away from the wall at a constant speed of 0.2 meters per second.



Goal: When the top of the ladder is at height h, calculate $\frac{dh}{dt}$, as a function of h.

- Given: $L^2 = h^2 + x^2$ and $\frac{dx}{dt} = 0.2$.
- Want: $\frac{dh}{dt} = ?$

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So $\boxed{\frac{dh}{dt} = -0.2\frac{\sqrt{L^2 - h^2}}{h}}$

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$$\frac{d}{dt}(L^2) = \frac{d}{dt}(h^2) + \frac{d}{dt}(x^2)$$

$$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt}$$

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Since $x = \sqrt{L^2 - h^2}$ and $\frac{dx}{dt} = 0.2$, we find

$$\boxed{\frac{dh}{dt} = -0.2\frac{\sqrt{L^2 - h^2}}{h}}$$

Sliding Ladder

Question: As the ladder is pulled away from the wall, what will happen to the top of the ladder?

1. It will slide all the way down the wall, staying in contact with the wall until it hits the ground.
2. It will slide down the wall until at some point it comes off the wall and freely falls to the ground.
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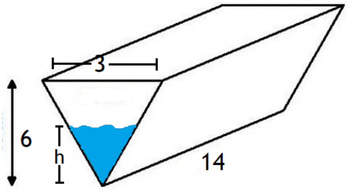
Suppose that option 1 is true. Then as $h \to 0$, $\frac{dh}{dt} \to -\infty$. In other words, the top of the ladder will have an infinite velocity as it hits the ground. This is physically impossible - gravity cannot accelerate an object to an infinite velocity in finite time!

Related Rates: Water Trough

Water flows out of the triangular trough shown at a constant rate of 2 dm^3/s. All numbers shown are measured in dm. Let h denote the height of the water in the trough.



Exercise: Calculate $\frac{dh}{dt}$, as a function of h.

Bonus slide - solution

Let V be the volume of water in the trough. Let x denote the width of the water in the trough. By similar triangles, $x = h/2$, so

$$V = 14 \cdot \frac{hx}{2} = 14 \cdot \frac{h^2}{4} = \frac{7}{2}h^2$$

Thus, $\frac{dV}{dt} = 7h\frac{dh}{dt}$. We are given that $\frac{dV}{dt} = 2$, and so $\boxed{\frac{dh}{dt} = \frac{2}{7h}}$.

Explicit vs. Implicit Functions

Explicit	Implicit
$y = \sqrt{x}$	$x^2 + y^2 - 25 = 0$
$y = \frac{x}{x^2+1}$	$x^2 + 3y^2 - xy - 11 = 0$
$y = f(x)$	$g(x, y) = 0$

- Every explicit function can be written as an implicit function, but not vice versa!
- For example, implicit functions usually do not pass the vertical line test.

The Derivative of an Implicit Function

$$x^2 + y^2 - 25 = 0$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(25) = 0$$

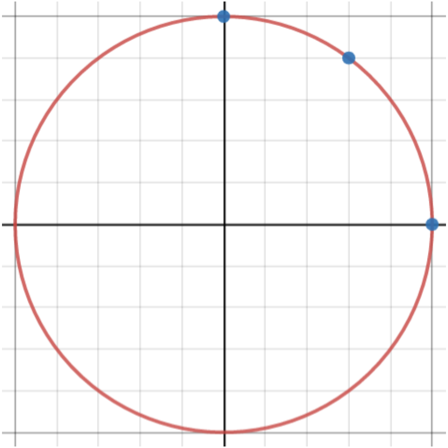
$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

The Derivative of an Implicit Function

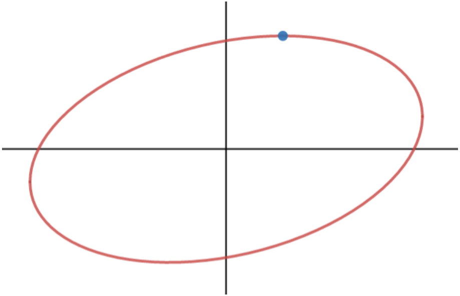
$$\boxed{dy/dx = -x/y}$$

Question: Calculate the slope of the tangent lines to the graph of $x^2 + y^2 - 25 = 0$ at the points $(0, 5), (3, 4)$, and $(5, 0)$.



The Derivative of an Implicit Function

Question: Shown is the graph of $x^2 + 3y^2 - xy - 11 = 0$. The blue point is the highest point on the graph. How would we calculate its coordinates?



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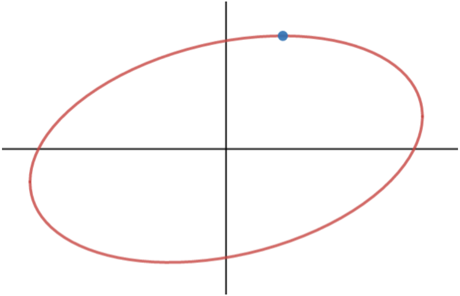


Strategy: At the point shown, $\frac{dy}{dx} = 0$, so let's calculate $\frac{dy}{dx}$ and set it equal to zero.

$$x^2 + 3y^2 - xy - 11 = 0$$

$$2x + 6y\frac{dy}{dx} - (x\frac{dy}{dx} + y) = 0$$

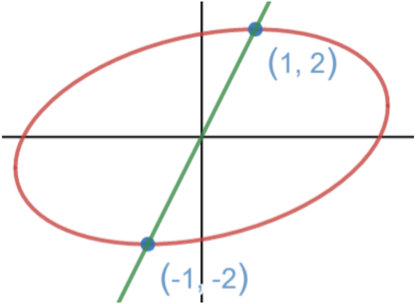
$$(6y - x)\frac{dy}{dx} + (2x - y) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{6y - x}}$$

$\frac{dy}{dx} = \frac{y-2x}{6y-x} = 0 \implies y = 2x$. Plug this back into our original equation $x^2 + 3y^2 - xy - 11 = 0$:

$$x^2 + 3(2x)^2 - x(2x) - 11 = 0$$

$$11x^2 - 11 = 0 \implies x = 1, y = 2 \text{ or } x = -1, y = -2$$



Recap

- The Chain Rule Applied: Related Rates
- Implicit Functions and Differentiation

Extra Practice

A conical cup with radius 3cm and height 10cm is leaking water at a rate of 1ml/sec. When the water is at height h, calculate $\frac{dh}{dt}$ in terms of h. (The volume of a cone with height h and radius r is $\frac{\pi r^2 h}{3}$.)